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DEFINITIONS OF THE DISCRIMINANT OF A RATIONAL INTEGRAL FUNCTION OF ONE VARIABLE.

By G. A. MILLER, University of Illinois.

Many students meet the term *discriminant* for the first time in the discussion of the solutions of the quadratic equation $ax^2 + bx + c = 0$. It is here commonly defined as the expression $b^2 - 4ac$ and it is usually noted that the vanishing of this expression is a necessary and sufficient condition that the given quadratic equation has equal roots. As the vanishing of this expression implies the vanishing of many other expressions and vice versa, it is clear that the given expression is by no means the only one whose vanishing is a necessary and sufficient condition that the quadratic equation under consideration has equal roots. Most of these other expressions might be said to be less simple but they include $4ac - b^2$ which appears to be just as simple as the expression defined above as the discriminant.

We emphasize these evident facts because the mathematical student is apt to meet in his more advanced work definitions of the term discriminant which are either vague or contradictory with his earliest use of the term. As an instance of the latter type of definitions we may cite the one given on page 96 of the *Theory of Equations* by F. Cajori, 1912. The discriminant of $f(x)$ is here defined as the resultant of $f(x)$ and $f'(x)$. If we find this resultant, when

$$f(x) = ax^2 + bx + c,$$

in accord with the illustration of the term resultant found on the same page, we obtain the following determinant:

$$\begin{vmatrix} a & b & c \\ 0 & 2a & b \\ 2a & b & 0 \end{vmatrix} = ab^2 - 4a^2c.$$

which is a times the discriminant of this quadratic as defined in the preceding paragraph.

In order to agree with the special case under consideration the discriminant of $f(x)$ might therefore be defined as the resultant of $f(x)$ and $f'(x)$ divided by the coefficient of the highest power of x in the rational integral function of x denoted by $f(x)$, provided we would adopt the definition of resultant implied in the illustration noted above. This definition, as well as the one given on page 196 of Bôcher's *Introduction to Higher Algebra*, 1907, is, however, not in accord with that found in some other standard works. In particular, these definitions do not coincide with the one found in the *Encyclopédie des Sciences Mathématiques*, tome 1, volume 2, page 75, but the resultants obtained according to these various definitions may differ only with respect to sign. This difference actually presents itself in the special case considered in the preceding paragraph, as can easily be verified.

We thus see that the reader may be considerably perplexed when he tries to harmonize the definitions of the term discriminant found in more advanced works with the earliest example of a discriminant with which he became acquainted, unless he is forewarned that the current definitions relating to this term are not in perfect agreement. The definition given on page 250 of Bôcher's *Introduction to Higher Algebra*, 1907, does not clear up the matter since it applies only to the case when the coefficient of the highest power of the unknown is unity. It may perhaps be taken for granted by many readers that the definition found in the large mathematical encyclopedias[1] will eventually be universally adopted. If this will be done we shall have to say that the discriminant of $ax^2 + bx + c$ is $4ac - b^2$ instead of $b^2 - 4ac$.

The question arises whether this change is desirable in view of the fact that $b^2 - 4ac$ appears under the radical sign in the common solution of the equation $ax^2 + bx + c = 0$ and hence it seems convenient to call it the discriminant of the quadratic equation rather than to say that this expression multiplied by -1 is the discriminant of the given equation. On the other hand, uniformity in definitions is so desirable that this slight convenience might well be sacrificed for the sake of avoiding the confusion caused by contradictory definitions of mathematical terms.

Another solution of this difficulty would be secured if mathematicians could agree on not following the large mathematical encyclopedias in this particular. In fact, such a well-known authority as the second edition of Pascal's *Repertorium der höheren Mathematik* has thrown its weight in favor of this movement by adopting a definition according to which the discriminant of $f(x)$ is the resultant of $f(x)$ and $f'(x)$ multiplied by

$$\frac{1}{a_0}(-1)^{n(n-1)/2},$$

[1] *Encyklopädie der Mathematischen Wissenschaften*, Vol. 1, p. 251; *Encyclopédie des Sciences Mathématiques*, tome 1, Vol. 2, pp. 97, 100.

where a_0 is the coefficient of the highest power of x in $f(x)$ and n is the degree of this function.[1] If this definition and the usual definition of resultant are adopted the discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$, which agrees with common usage in our textbooks.

When J. J. Sylvester first used the term discriminant as a technical mathematical term, *Philosophical Magazine*, 1851, page 406, he added the following explanatory remark: "Discriminant because it affords the discrimen or test for ascertaining whether or not equal factors enter into a function of two variables, or more generally, of the existence or otherwise of multiple points in the locus represented or characterized by an algebraical function." Several authors have used somewhat similar explanatory words as a definition. Thus we find in the *Theory of Equations* by F. Cajori, 1912, page 96, "The discriminant of an equation $f(x) = 0$ may be otherwise defined as the simplest function of the coefficients, or of the roots, whose vanishing signifies that the equation has equal roots." Such a definition is clearly too vague to determine a definite expression as the discriminant of a given function. This is also true of such definitions as the following: "The discriminant of an equation involving a single unknown is the simplest function of the coefficients in a rational integral form, whose vanishing expresses the condition for equal roots," *Theory of Equations* by Burnside and Panton, volume 2, 1904, page 83; cf. *Theory of Equations* by F. Cajori, 1912, page 97.

These observations may suffice to forewarn the young mathematician in regard to the different meanings assigned to common mathematical terms by various authors. Although the term discriminant is sixty-seven years old and has found a place in many elementary textbooks it is still being defined differently by eminent authorities, so that it would even now be difficult to determine by the process of comparing authorities what expression should be called the discriminant of the general quadratic equation in one unknown. In addition to the noted difficulties in the way of such a determination we may add that on page 315 of the first volume of Borel's *Die Elemente der Mathematik* translated by P. Stäckel, 1908, it is stated that

$$\frac{b^2 - 4ac}{4}$$

is sometimes called the discriminant of the equation $ax^2 + bx + c = 0$.

It is a very singular fact that the definition of the discriminant of $f(x)$ contained in what are now universally regarded as our foremost works of reference, viz., the large mathematical encyclopedias in course of publication, is not in accord with the common textbook use of this term in the special case when $f(x) = ax^2 + bx + c$. The main objects of this note are to direct more general attention to this fact and to save the beginner from the perplexities into which

[1] Pascal's *Repertorium der höheren Mathematik*, Vol. 1 (1910), p. 274. In the 1900 edition of this work, p. 88, the definition of the discriminant of $f(x)$ agrees with that found in the large mathematical encyclopedias.

he is naturally led by definitions found in some of our reliable American textbooks. In a previous note published in this MONTHLY, volume 24, 1917, page 106, we directed attention to desirable changes in the definitions of the term discriminant found in two of our standard dictionaries and in the *International Encyclopedia.*

In view of the fact that such eminent authorities give different definitions of the term discriminant of $f(x)$ it may appear almost presumptuous on the part of the present writer to express an opinion in regard to what appears to him as the most desirable definition to adopt. The advantages of uniformity along this line impel him, however, to say that he would adopt the definition found in the second edition of Pascal's *Repertorium* noted above, and hence he would not follow the large mathematical encyclopedias in this particular case. He would, on the other hand, follow these encyclopedias as regards the definition of the resultant of two functions of a single variable, and hence he would digress also in this instance from the definition adopted in Bôcher's *Algebra* cited above. These conclusions are based mainly on what appears to the writer as the most feasible steps towards securing uniformity.

Dr. A. J. Kempner recently directed my attention to the following definition: "By the *resultant* of two equations $f(x) = 0$ and $\phi(x) = 0$ is meant that integral function of the coefficients of $f(x)$ and $\phi(x)$ whose vanishing is the necessary and sufficient condition that $f(x) = 0$ and $\phi(x) = 0$ have a common root"; *College Algebra* by H. B. Fine, 1904, page 512. The fact that this definition appears in one of our most advanced American algebras may justify a reference to it here, especially since the definitions of the terms discriminant and resultant are so closely related. It is difficult to see how such a vague definition can fail to embarrass the serious student.

In this connection it may be of interest to refer to another term (*real curve*) which is used in college mathematics with a meaning differing from that assigned to it in some of our best reference books. To illustrate this fact we may note that in the *Encyclopédie des Sciences Mathématiques,* tome 3, volume 3, page 261, it is stated that "a real curve need not contain any real point; as happens, for instance, in the case of the real conic $x^2 + y^2 + 1 = 0$." On the contrary, it is customary to call such a conic in our textbooks an *imaginary* curve notwithstanding the fact that all the coefficients of its equation are real, and this usage seems to be also in accord with the terminology employed on page 71 of the volume just mentioned. The use of the term *real curve* for an imaginary ellipse in analytic geometry does not appear desirable, and hence it is the more questionable whether this use should be sanctioned in more advanced subjects.